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Abstract

This study proposes a model of non-unitary time discounting and examines its welfare implications. A key feature of our model lies in the disparity of time discounting between multiple distinct goods, which induces an individual's preference reversals even though she normally discounts her future utilities for each good. After characterizing the time-consistent decision-making by such an individual in a general setting, we compare welfare achieved in the market economy and welfare in the planner's allocation from the perspective of all selves across time. Under certain situations, the selves in early periods strictly prefer the social planner's allocation, whereas the selves in future periods strictly prefer the market equilibrium. Therefore, the welfare implications of our model are quite different from those in the canonical discounting model and in models of other time-inconsistent preferences.

JEL classification: E21; H21; O41

Keywords: Non-unitary time discounting; Time inconsistency; Time-consistent tax policy.

1 Introduction

Father: "Could you mow the yard tomorrow instead of playing football? After completing the job, I will give you $20."

Son: "Really? I will. Then, I can buy a new computer game!"

Tomorrow has come.

Father: "Why are you going out to play football? Mow the yard! You promised yesterday, didn't you?"

Son: "Sorry Dad. I no longer think $20 is enough for the job."

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Why did the boy break his promise? Is it because he is a liar? Of course, there are a number of possible answers to this question. One possibility, suggested by a large body of experimental evidence, is that preference reversals frequently occur over time in people's decision-making. As such, it could be that the boy first regarded \$20 (or purchasing a new game) as preferable, but by the following day, preferred his leisure activity.

Although only one among many hypothetical answers, this possibility becomes convincing once we consider the *domain effect*, or *domain independence*, often referred to in the experimental psychology literature.[1] The domain effect emerges when the discount rates (or factors) differ depending on their domains. In the abovementioned example, the domain effect emerges if the boy discounts the utility from the monetary reward (\$20) and that from enjoying the leisure activity (football) differently. For expositional convenience, let R denote the utility from the monetary reward and F denote the utility from the leisure activity. We assume $R < F$; that is, the boy will never mow the yard if asked to do so right now. Next, suppose that, on the first day, he evaluates the utility from receiving \$20 as $\beta_1 R$, and that from playing football as $\beta_2 F$, where $\beta_1 \in (0, 1)$ and $\beta_2 \in (0, 1)$ are the discount factors *specific* to the monetary reward and leisure, respectively. Then, if the boy discounts enjoying leisure steeply enough such that $\beta_1 R > \beta_2 F$, he will accept his father's job offer on the first day.

Hereafter, we refer to such domain-specific discounting as non-unitary discounting. If an individual discounts her future utilities in a non-unitary way, this can make her decisions time inconsistent. There has been a recent upsurge of interest in models of time-inconsistent preferences, as pioneered by Strotz (1955) and Pollak (1968). In this context, the individual's decision-making process is formulated as a dynamic non-cooperative game played by her different selves across time, where the current self is aware that her preferences might change in future, and takes this into account when making the current decision.[2] However, much of the literature focuses on a class of quasi-hyperbolic discounting proposed by Phelps and Pollak (1968), and popularized by Laibson (1997). Therefore, the purpose of this study is to develop a simple dynamic theory of non-unitary discounting.

In this study, we develop a simple model of non-unitary time discounting and pursue its welfare implications. As in Hori and Futagami (2019), an individual discounts her one-period utility functions of consumption and leisure differently. As the boy does in the earlier example, the individual changes her mind about the relative importance of consumption and leisure as time progresses. Within this framework, we compare welfare achieved in the market economy from welfare in the planner's allocation from the perspective of *all* selves across time. The results are no longer straightforward, because as a result of a lack of commitment, each self of the social planner is also involved in strategic interactions with her other selves. In fact, in their model with quasi-hyperbolic discounting, Krusell et al. (2002) show that the allocation in the market economy surprisingly attains strictly higher welfare than that in the planning allocation. Hiraguchi (2014) extends Krusell et al. (2002) to a general model of non-constant discounting, including the original as a special case, and shows that their result is robust. At the same

[1] For example, based on her experimental studies, Chapman (1996) notes that discount rates may be specific to money and health status. For an excellent discussion on the inconsistency of intertemporal choices due to time discounting, see Frederick et al. (2002).

[2] For example, see Peleg and Yaari (1973) and Goldman (1980) for the game-theoretic foundations of the solution concepts in Strotz (1955) and Pollak (1968).

time, a welfare comparison between the competitive and planning economies gives rise to the following problem. In order to correctly identify which achieves higher welfare in each period, we must control the difference in the dynamics of the state variables between the two economies. In other words, we cannot evaluate which of the economies performs better if we focus only on their overall paths.[3]

Then, we conduct a welfare comparison in two distinct ways. First, we consider the hypothetical situation in which, in an arbitrarily given period, a self faces the same value of a state variable in both economies. It is shown that welfare in the social planning case is always strictly higher than that in the market economy. This means that welfare improvement is always possible from the realized allocation in the market economy, which contrasts sharply with the findings of Krusell et al. (2002). Second, we undertake a welfare comparison between the overall paths of the two economies. We show that whether the planning allocation is more Pareto efficient than the allocation in the market economy depends on the relative degree of impatience. The following two cases arise. If the individual discounts future leisure more steeply than she does future consumption, the planning allocation is preferable to the laissez-faire allocation for all her selves. However, if the reverse is the case, they are not Pareto ranked. In this case, we show that there is a unique threshold period before which any selves strictly prefer the planning allocation. However, after this period, they strictly prefer the laissez-faire allocation. This means that the allocation in the market equilibrium may achieve a more desirable outcome than the social planner does for the selves in later periods.

As already stated, the most closely related literature to our study is the set of studies on time-inconsistent preferences resulting from non-geometric discounting. However, our model is also related to a class of preferences exhibiting temptations. Among others, Banerjee and Mullainthan (2010) consider a two-period, many-good economy, and classify the goods into two types. The first is a standard good, the consumption of which in both periods yields the individual's lifetime utility in period 1. The second is a "temptation good," the consumption of which in period 2 is not valued in period 1, but yields utility once period 2 has arrived. In their two-period model, temptation goods are interpreted as those with a discount factor of 0. If we set $\beta_2 = 0$ in the example at the beginning of the introduction, playing football is a temptation good for the boy. Thus, our model of non-unitary discounting is closely related to their notion of temptation.[4]

The remainder of this paper is organized as follows. Section 2 gives the illustrative example of non-unitary discounting, and explains why the time-inconsistency problem arises. It also provides the Euler equation in this model. Section 3 then extends the framework to a dynamic general equilibrium model and characterizes the market equilibrium. Section 4 compares welfare in the market economy to that in the social planner's allocation. It also shows the existence of the time-consistent policies by the benevolent government. Section 5 concludes.

[3] In Subsection 4.4 (p. 56) of their paper, Krusell et al. (2002) make the same argument.

[4] In other words, in the model of Banerjee and Mullainthan (2010), time inconsistency occurs. By contrast, as is well known, Gul and Pesendorfer (2001) propose a utility function (and give its axiomatic foundations) that exhibits temptation, but that is free from time inconsistency.

2 Preliminary

2.1 Time-inconsistency Problem due to Non-unitary Discounting

We start with a consumer's optimization in a two-good model, provided that all prices are exogenously constant. Time is discrete and indexed by $t = 0, 1, 2, \ldots$ There are two distinct goods, c and x. Preferences of an infinitely lived consumer in period t are given as the following utility function:

$$\begin{aligned} U_t &= u(c_t) + \beta_c u(c_{t+1}) + \beta_c^2 u(c_{t+2}) + \beta_c^3 u(c_{t+3}) + \cdots \\ &\quad + v(x_t) + \beta_x v(x_{t+1}) + \beta_x^2 v(x_{t+2}) + \beta_x^3 v(x_{t+3}) + \cdots \\ &= \sum_{t'=t}^{\infty} \left(\beta_c^{t'-t} u(c_{t'}) + \beta_x^{t'-t} v(x_{t'}) \right), \end{aligned} \tag{1}$$

where $u(c)$ and $v(x)$ denote the one-period utility functions from consuming c and x, respectively. Both u and v are twice differentiable and satisfy $u'(c) > 0, v'(x) > 0, u''(c) < 0$, and $v''(x) < 0$. Parameters $\beta_c \in (0, 1)$ and $\beta_x \in (0, 1)$ are the subjective discount factors for goods c and x, respectively. If $\beta_c = \beta_x$ always holds, then the utility function (1) is a canonical one. In this study, we allow the case of $\beta_c \neq \beta_x$, that is, the consumer discounts utility from different sources at different rates.

As shown by Ubfal (2016) and Hori and Futagami (2019), the time-inconsistency problem arises when $\beta_c \neq \beta_x$. This is because such good-specific discounting induces an *intertemporal* variation in an *intratemporal* marginal rate of substitution (MRS). Assume two dates, t and $T(> t)$. From (1), we obtain the following relationship for all $c_T > 0$ and $x_T > 0$:

$$\underbrace{\left(\frac{\beta_x}{\beta_c}\right)^{T-t} \frac{v'(x_T)}{u'(c_T)}}_{\equiv -dc_T/dx_T|_{dU_t=0}} \gtreqless \underbrace{\frac{v'(x_T)}{u'(c_T)}}_{\equiv -dc_T/dx_T|_{dU_T=0}},$$

if and only if $\beta_x \gtreqless \beta_c$. The left-hand side is the MRS between c_T and x_T evaluated in period t, whereas the right-hand side is the same MRS evaluated in period T. Thus, as the evaluation date is updated, the consumer changes her mind about the relative importance between the two goods when $\beta_c \neq \beta_x$. For example, suppose $\beta_c > \beta_x$: consumption of good x is less postponable than good c from the perspective of the current consumer. This means that the current consumer does not consider future consumption of good x to be especially important. However, the above relationship shows that as the evaluation date t approaches the execution date T, consumption of good x becomes more attractive than under her original plan.

Note that the driving force of time inconsistency is significantly distinguished from the Phelps–Pollak–Laibson preferences of quasi-hyperbolic discounting. In their preferences, the driving force is a time variation of the *intertemporal* MRS between consumption of a single good in two adjacent periods.

2.2 Intrapersonal Game and the Euler Equation

Then, how does such a preference reversal influence a consumer's intertemporal behavior? We follow Phelps–Pollak–Laibson to formulate the consumer's problem as a dynamic intrapersonal game, where the consumer is composed of a sequence of their distinct "selves" indexed by period t. In their continuous-time model of non-unitary discounting, Hori and Futagami (2019) derive the (generalized) Euler equation under the situation in which a consumer in a period cannot commit to her future selves' actions. To obtain clear-cut results, however, Hori and Futagami (2019) follow the calculation procedure developed by Barro (1999) and thus, specify the functional form of instantaneous utilities from the beginning. By contrast, our discrete-time framework enables us to derive the Euler equation with general functional forms of u and v.

The consumer's flow budget constraint is given by

$$a_{t+1} = Ra_t - c_t - px_t, \tag{2}$$

where a_t denotes the asset holding in period t, $R > 1$ is the gross interest rate, and $p > 0$ is the price of good x. To focus on a single individual's saving behavior, we assume that prices are exogenous and ignore wage income here, both of which are relaxed when we introduce non-unitary discounting into a dynamic general equilibrium model in Section 3.

Throughout the study, we assume that each self is sophisticated so that she takes her next self's decision-making into account. Specifically, the self in period t rationally expects her next self's decisions to be given by $c_{t+1} = \phi_c(a_{t+1})$ and $x_{t+1} = \phi_x(a_{t+1})$, where functions ϕ_c and ϕ_x are unknown still to be solved. Once they are given, a_{t+2} is accordingly given by $a_{t+2} = g(a_{t+1})$, where $g(a)$ is given by $g(a) \equiv Ra - \phi_c(a) - p\phi_x(a)$ from the budget constraint (2). Thus, the self in period t decides the level of assets in the next period a_{t+1} with the knowledge that it affects the subsequent self's actions.[5]

The optimization problem of the self in a period with her assets given by a is formulated as

$$V(a) = \max_{c,x,a'} \left\{ u(c) + v(x) + \beta_c V_c(a') + \beta_x V_x(a') \;\middle|\; a' = Ra - c - px \right\},$$

where V is the value function associated with this problem. In addition, functions V_c and V_x are defined as the solutions to the following functional equations:

$$V_c(a') = u(\phi_c(a')) + \beta_c V_c(g(a')), \quad V_x(a') = v(\phi_x(a')) + \beta_x V_x(g(a')).$$

Concerning the intratemporal decision, we obtain $v'(x_t)/u'(c_t) = p$, which means simply that the MRS between c_t and x_t equals the relative price p. Concerning the dynamic decision, we obtain the following proposition:

Proposition 1. *The Euler equation in this model is given by*

$$u'(c_t) = \beta_c u'(c_{t+1})R + (\beta_x - \beta_c)\frac{dV_x(a_{t+1})}{da_{t+1}},$$

or equivalently

$$v'(x_t) = \beta_x v'(x_{t+1})R + (\beta_c - \beta_x)p\frac{dV_c(a_{t+1})}{da_{t+1}}.$$

Proof. See the Appendix. □

If $\beta_x = \beta_c$, the second term on the right-hand side does not appear, leading to the canonical Euler equation. By contrast, if $\beta_x \neq \beta_c$, the second term additionally provides the marginal cost (or reward) of

[5] Throughout the study, we focus on the case in which each self employs Markov strategies whereby she makes a decision based only on the state variables, in this case, her assets.

savings. The intuition behind the emergence of this additional term is explained as follows. Suppose that the self in period t marginally increases her savings at the cost of marginal disutility $u'(c_t)$. In response, the self in period $t+1$ changes consumption for both goods by $d\phi_c(a_{t+1})/da_{t+1}$ and $d\phi_x(a_{t+1})/da_{t+1}$ units. The resulting marginal utility for this self is given by

$$\frac{d}{da_{t+1}}\left[V_c(a_{t+1}) + V_x(a_{t+1})\right] = u'(c_{t+1})R.$$

The proof is given in the Appendix. However, the self in period t evaluates $dV_x(a_{t+1})/da_{t+1}$ by its β_x times, not by β_c. Then, the marginal utility for the self in period t is $\beta_c \frac{dV_c(a_{t+1})}{da_{t+1}} + \beta_x \frac{dV_x(a_{t+1})}{da_{t+1}}$, which is less (more) than $\beta_c u'(c_{t+1})R$ when $\beta_x < (>)\ \beta_c$. In other words, the next self's consumption additionally provides the current self with the marginal cost (reward) of savings.

3 Dynamic General Equilibrium Model

As shown in the previous section, a key feature of our model lies in the disparity of time discounting between multiple distinct goods. To obtain macroeconomic implications of this aspect and pursue its welfare implications, we now extend the model to a dynamic general equilibrium model. Following Hori and Futagami (2019), we assume that households differently discount their utility from consumption of a good and their utility from consumption of leisure.

3.1 Production

A final good is used for consumption and investment. The production function takes a Cobb–Douglas form, $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$, where Y, K, and L denote the amount of output, demand for capital, and demand for labor, respectively. The parameter $A > 0$ is the level of total factor productivity and $\alpha \in (0,1)$ is a constant that specifies the share of capital income in total output. Let $X_t \equiv K_t/L_t$ denote the ratio of aggregate demand for capital to that for labor. Then, perfect competition results in

$$r_t = r(X_t) \equiv A\alpha X_t^{\alpha-1}, \quad w_t = w(X_t) \equiv A(1-\alpha)X_t^{\alpha}. \tag{3}$$

3.2 Households

There is a continuum of homogeneous households with unit mass. Each household (denoted as "she" in this paper) is endowed with one unit of time, and now derives her utility from consumption of the good and leisure time. Letting l_t denote her labor supply, her consumption of leisure is $1 - l_t$. The utility function is now given by $U_t = \sum_{t'=t}^{\infty}[(\beta_c^{t'-t}u(c_{t'}) + \beta_l^{t'-t}v(1-l_{t'})]$, where $\beta_c \in (0,1)$ and $\beta_l \in (0,1)$ are the discount factors applied to her consumption of the good and leisure time, respectively.

Letting k_t denote the amount of capital held by the individual in period t, the flow budget constraint is given by

$$k_{t+1} = R_t k_t + w_t l_t - c_t,$$

where $R_t \equiv r_t + 1 - \delta$ and $\delta \in [0,1]$ is the depreciation rate of capital. The aggregate variable X_t is taken as given by each individual.

Here, let us redefine a_t as

$$a_t \equiv k_t + \sum_{t'=t}^{\infty} \frac{w_{t'}}{\prod_{\nu=t}^{t'} R_\nu},$$

meaning that a_t is now total wealth, which is given by the sum of capital and human wealth. Under the additional condition that $\lim_{T\to\infty}(\prod_{t=0}^{T} R_t)^{-1} a_{T+1} = 0$, which is the transversality condition and indeed holds in equilibrium, the above budget constraint is rewritten as[6]

$$a_{t+1} = R_t a_t - c_t - w_t(1-l_t). \tag{4}$$

Once we replace p with w_t and x_t with $1-l_t$, we find that each self faces the same problem as identified in the previous section, except that the prices R_t and w_t are now time-varying. Therefore, we have to formulate each self's expectation about how these values evolve over time. The self in period t rationally expects the law of motion of X_t as

$$X_{t+1} = G(X_t), \tag{5}$$

where function G is a function to be solved. Since $R_t = R(X_t) \equiv r(X_t) + 1 - \delta$ and $w_t = w(X_t)$, equation (5) captures the individual's expectation for the prices in the next period.

Therefore, the optimization problem is recursively formulated as

$$V(a, X) = \max_{c,l,a'} \{u(c) + v(1-l) + \beta_c V_c(a', G(X)) + \beta_l V_l(a', G(X))\}, \tag{6}$$

subject to equation (4). In this problem, functions $V_c(a, X)$ and $V_l(a, X)$ on the right-hand side are defined in the same manner as the previous section:

$$V_c(a, X) = u(\phi_c(a, X)) + \beta_c V_c(g(a, X), G(X)),$$
$$V_l(a, X) = v(1-\phi_l(a, X)) + \beta_l V_l(g(a, X), G(X)),$$

where $\phi_c(a, X)$ and $\phi_l(a, X)$ are the policy functions for c and l. Function g is defined as $g(a, X) \equiv R(X)a - \phi_c(a, X) - w(X)(1-\phi_l(a, X))$ which gives the level of assets in the next period. Following the same calculation procedure as Proposition 1, we obtain the following equations:

$$\frac{v'(1-l_t)}{u'(c_t)} = w(X_t), \tag{7}$$

$$u'(c_t) = \beta_c u'(c_{t+1}) R(G(X_t)) + (\beta_l - \beta_c) \frac{dV_l(a_{t+1}, G(X_t))}{da_{t+1}}. \tag{8}$$

3.3 Equilibrium

The Euler equation (8) is very informative when we grasp how and why the time-inconsistency problem arises in this class of preferences. At the same time, however, note that the unknown function V_l exists in this equation, meaning that we cannot analytically characterize the equilibrium unless we specify the one-period utility functions to obtain the functional form of V_l. Therefore, we specify u and v as

$$u(c) = \ln c, \quad v(1-l) = \zeta \ln(1-l), \quad \zeta > 0,$$

respectively. Under this specification, we first show the following lemma:

[6]Using the definition of a_t and $\lim_{T\to\infty}(\prod_{t=0}^{T} R_t)^{-1} a_{T+1} = 0$, the intertemporal budget constraint is given by $a_t = \sum_{t'=t}^{\infty}(\prod_{\nu=t}^{t'} R_\nu)^{-1}(c_{t'} + w_{t'}(1-l_{t'}))$ for all $t = 0, 1, 2\ldots$ Then, we obtain $R_t a_t = a_{t+1} + c_t + w_t(1-l_t)$.

Lemma 1. *Given a_t and X_t, a_{t+1} is given by $a_{t+1} = g(a_t, X_t) = \gamma R(X_t) a_t$, where γ is defined as*

$$\gamma \equiv \frac{\beta_c + \lambda}{1 + \lambda} \in (0, 1),$$

and λ is

$$\lambda \equiv \frac{\zeta}{1 + \zeta} \frac{\beta_l - \beta_c}{1 - \beta_l}.$$

Proof. See the Appendix. □

Therefore, λ gives the criterion for whether the preferences exhibit non-unitary discounting, since it deviates from zero if and only if $\beta_c \neq \beta_l$. To analytically obtain the equilibrium, we must also assume full depreciation of capital:

Assumption 1. $\delta = 1$.

This assumption is restrictive, but is made in common with many existing studies that examine time-inconsistent preferences for the purpose of analytical characterization of the equilibrium (Krusell et al., 2002; Hiraguchi, 2014, 2016). We show that even under such a restrictive situation, welfare implications of our model dramatically change when λ takes a non-zero value.

The market-clearing conditions are given by $K_t = k_t$ and $L_t = l_t$, which jointly mean $X_t = k_t/l_t$. We then derive the competitive equilibrium by use of a simple "guess and verify" method. We first guess that the equilibrium labor supply is constant over time, $l_t = l^{eqm}$. We then guess that the functional form of G is

$$G(X) = s^{eqm} A X^{\alpha}.$$

In other words, we guess that the saving rate is constant over time. In addition, we guess that the saving rate s^{eqm} satisfies

$$s^{eqm} < \alpha,$$

which is verified in the equilibrium.

We now derive s^{eqm} and l^{eqm}, and verify that these are indeed constant over time. Based on Assumption 1, we can rewrite human wealth as the following simple expression:

$$\sum_{t'=t}^{\infty} \frac{w_{t'}}{\prod_{\nu=t}^{t'} r_{\nu}} = \frac{1 - \alpha}{(\alpha - s^{eqm}) l^{eqm}} k_t, \tag{9}$$

where the detailed derivation process is given in the Appendix. Then, equation (9) gives total wealth a_t as

$$a_t = a(k_t) \equiv \left[1 + \frac{1 - \alpha}{(\alpha - s^{eqm}) l^{eqm}}\right] k_t.$$

Substituting this result, equation (3), and $X_t = k_t / l^{eqm}$ into equation $a_{t+1} = \gamma R(X_t) a_t$ shown in Lemma 1 yields the following dynamic equation of k_t:

$$k_{t+1} = \gamma \alpha A k_t^{\alpha} (l^{eqm})^{1-\alpha}, \tag{10}$$

meaning that

$$s^{eqm} = \gamma \alpha \in (0, \alpha).$$

The equilibrium consumption is accordingly given by $(1-s^{eqm})A(k_t)^\alpha(l^{eqm})^{1-\alpha}$. Finally, substituting this result into equation (7), we obtain l^{eqm} as

$$l^{eqm} = \frac{1-\alpha}{1-\alpha+\zeta(1-\gamma\alpha)} \in (0,1), \tag{11}$$

which is indeed constant over time.

Given the initial condition $k_0 > 0$, the equilibrium sequence of capital is determined from (10). Let $\{k_t^{eqm}\}$ denote the sequence. Once it is determined, the sequence of all other variables is determined accordingly.

Proposition 2. *There is a unique competitive equilibrium path in this model.*

By incorporating quasi-hyperbolic discounting into a simple neoclassical growth model and assuming a logarithmic utility function, Krusell et al. (2002) show that observational equivalence holds between their model and the standard geometric discounting model.[7] Barro (1999) shows the same property in his continuous-time model of non-constant rate of time preferences. Such observational equivalence also holds between our non-unitary discounting model and the standard discounting model. Consider an economy in the same environment as ours, except that the representative individual's preference is given by $\sum_{t'=t}^{\infty}\gamma^t[\ln c_t + \zeta\ln(1-l_t)]$. The equilibrium conditions in such a model are given by (10) and (11). At first glance, this result appears to show that the standard model or the model of quasi-hyperbolic discounting can replicate all our findings. This conjecture is not correct because, as we show in the next section, our model yields welfare implications that differ markedly from these preferences.

4 Welfare Implications

In this section, we show that the competitive equilibrium characterized in the previous section generates inefficiencies. For this purpose, we first derive the allocation by the social planner who can directly affect the resource constraint by her decisions. The social planner's preferences are the same as those of the individual in Section 3, and she cannot commit to her future selves' decisions. Thus, as in the case of the market economy, distinct selves of the planner play an intrapersonal game.

The optimization problem of the planner in a period is given by

$$V^{sp}(k) = \max_{k',l}\left\{\ln(Ak^\alpha l^{1-\alpha}-k') + \zeta\ln(1-l) + \beta_c V_c^{sp}(k') + \beta_l V_l^{sp}(k')\right\},$$

where $V^{sp}(k)$ is the value function of the household when the social planner directly designs the allocation without a market mechanism. Functions $V_c^{sp}(k)$ and $V_l^{sp}(k)$ are given by the following functional equations:

$$\begin{aligned}
V_c^{sp}(k) &= \ln\left[Ak^\alpha(\phi_l^{sp}(k))^{1-\alpha} - g^{sp}(k)\right] + \beta_c V_c^{sp}(g^{sp}(k)),\\
V_l^{sp}(k) &= \zeta\ln(1-\phi_l^{sp}(k)) + \beta_l V_l^{sp}(g^{sp}(k)).
\end{aligned}$$

[7] See Proposition 2 of their paper.

Lemma 2. *In the social planner's allocation, $l_t = l^{sp}$ and $k_{t+1} = s^{sp} A k_t^{\alpha} (l^{sp})^{1-\alpha}$, where*

$$s^{sp} = \beta_c \alpha,$$
$$l^{sp} = \frac{1-\alpha}{1-\alpha+\zeta(1-\beta_c\alpha)}.$$

Proof. See the Appendix. □

Hereafter, let $\{k_t^{sp}\}$ denote the sequence of capital realized in the social planner's allocation.

4.1 Market Equilibrium versus Social Planning

We introduce the following function:

$$V^{eqm}(k) \equiv V(a(k), k/l^{eqm}),$$

where function $V(\cdot,\cdot)$ on the right-hand side is the value function of the household in the market economy, defined in equation (6). In this study, we compare welfare achieved in the market economy with that in the planner's allocation from the perspective of *all* selves. In other words, we compare $V^{eqm}(k_t^{eqm})$ and $V^{sp}(k_t^{sp})$ for all $t = 0, 1, 2, \ldots$

If $\beta_c = \beta_l$, the value of parameter γ becomes equal to β_c and then $s^{eqm} = s^{sp}$ and $l^{eqm} = l^{sp}$. This yields the standard result that the competitive equilibrium achieves the socially optimal allocation. By contrast, if $\beta_c \neq \beta_l$, this theorem no longer holds.

Lemma 3. *Suppose that $\beta_c > (<)\ \beta_l$. Then, $s^{eqm} < (>)\ s^{sp}$ and $l^{eqm} < (>)\ l^{sp}$.*

Proof. As shown in Lemma 2, both s^{sp} and l^{sp} are independent of β_l. Meanwhile, the equilibrium saving rate s^{eqm} is given by $\gamma\alpha$, which is strictly increasing with respect to β_l from the definition of γ. The equilibrium labor supply l^{eqm} is given by equation (11), which is also strictly increasing with respect to β_l. □

Lemma 3 states that if consumption of a good is more (less) postponable than leisure is, then the saving rate in the market equilibrium is lower (higher) than under the social planner's allocation. It also states that the equilibrium labor supply becomes low (high).

Welfare Comparison in the Initial Period: The difference in the household's saving- and working behaviors between the two economies induces the difference in the value function between them: V^j, $j \in \{eqm, sp\}$. We first focus on this difference and obtain the following proposition.

Proposition 3. *Suppose that $\beta_c \neq \beta_l$. Then, $V^{eqm}(k) < V^{sp}(k)$ always holds for all $k > 0$.*

Proof. See the Appendix. □

This proposition states that given her capital holding k, a self in any period strictly prefers the social planning allocation than the market equilibrium allocation. Indeed, since the initial stock of capital is the same in both economies, Proposition 3 states that from the perspective of the initial self, social planning always achieves higher welfare than the market economy does. This result contrasts sharply with the findings of Krusell et al. (2002), whose quasi-hyperbolic discounting model shows that the market equilibrium always performs better than the planning economy.

Welfare Comparison in the Subsequent Periods: Then, does this result apply to the other selves? The difference in the household's behavior also induces the difference in the level of capital in the subsequent periods. In this model, we can explicitly solve the sequence of capital in both economies:

$$\begin{aligned} k_t^j &= \mathcal{K}(s^j, l^j, t) \\ &\equiv \exp\left\{\alpha^t \ln k_0 + \frac{1-\alpha^t}{1-\alpha} \ln\left[s^j A(l^j)^{1-\alpha}\right]\right\}, \quad j \in \{eqm, sp\}. \end{aligned} \tag{12}$$

As is apparent from (12), $\mathcal{K}(s, l, t)$ is strictly increasing with respect to both s and l. Then, Lemma 3 yields the following lemma.

Lemma 4. *Suppose that $\beta_c > (<)\ \beta_l$. Then, given $k_0^{eqm} = k_0^{sp} = k_0 > 0$, $k_t^{eqm} < (>)\ k_t^{sp}$ for all $t = 1, 2, \ldots$*

In addition, we can show that

Lemma 5. *$V^j(k)$ is a strictly increasing function for all $j \in \{eqm, sp\}$.*

Proof. See the Appendix. □

Then, we arrive at the following proposition, showing that in the case of $\beta_c > \beta_l$, the allocation in the market economy is Pareto dominated by that of the social planning from the perspective of all selves across time.

Proposition 4. *Suppose that $\beta_c > \beta_l$. Then, given $k_0^{eqm} = k_0^{sp} = k_0 > 0$, $V^{eqm}(k_t^{eqm}) < V^{sp}(k_t^{sp})$ for all $t = 0, 1, 2 \ldots$*

Proof. Lemmas 4, 5, and Proposition 3 in the main body jointly show this proposition. □

Next, consider the case of $\beta_l > \beta_c$. In this case, comparison of $V^{eqm}(k_t^{eqm})$ and $V^{sp}(k_t^{sp})$ is not straightforward. In the Appendix, we provide the proof for the following proposition.

Proposition 5. *Suppose that $\beta_l > \beta_c$. Then, there exists a unique $\overline{\beta}_l \in (\beta_c, 1)$. When $\beta_c < \beta_l < \overline{\beta}_l$, there exists a unique $T^* > 0$, such that $V^{eqm}(k_t^{eqm}) > V^{sp}(k_t^{sp})$ if and only if $t \geq T^*$.*

Proof. See the Appendix. □

Proposition 5 states that the equilibrium allocation can achieve a more desirable outcome than the social planner's allocation for the selves in later periods. The intuition behind this result is explained as follows. As stated in Lemma 3, the household cannot help saving excessively in the market economy when $\beta_l > \beta_c$. Indeed, as shown in Proposition 3, this induces a welfare loss for the individual. However, such a decision by the household is favorable for her future selves, because the assets of these selves increase rapidly.

Based on the above results, can we then accept that surprising conclusion that, in the long run, a laissez-faire environment performs a better job than the social planner does? We must be cautious when answering this question, To clarify this point, we focus on the case of $\beta_c < \beta_l < \overline{\beta}_l$. Then, from Propositions 3 and 5, there exists a period $t \geq T^*$, such that the following two inequalities are satisfied simultaneously:

$$V^{sp}(k_t^{sp}) < V^{eqm}(k_t^{eqm}) < V^{sp}(k_t^{eqm}).$$

The first inequality shows that the market equilibrium is more desirable than social planning for the self in this period, because she can obtain more assets in the market economy. However, the second inequality also shows that, given k_t^{eqm}, the self strictly prefers the allocation by the social planner. Thus, the market equilibrium is suboptimal and welfare improvement is always possible from its realized allocation.

4.2 Implications of Tax Policies

Finally, we now introduce government activity to the market economy. We assume that the government imposes taxes on individuals' wage income, capital income, and savings. Letting $\tau_{r,t} \in (0,1)$, $\tau_{w,t} \in (0,1)$, and $\tau_{i,t} \geq 0$ denote the rates of these taxes, the household's budget constraint now becomes

$$(1+\tau_{i,t})k_{t+1} = (1-\tau_{r,t})r_t k_t + (1-\tau_{w,t})w_t l_t - c. \tag{13}$$

We assume that there is no government expenditure and the government's budget must be balanced in each period. The government's budget constraint is given by

$$\tau_{r,t} r_t K_t + \tau_{w,t} w_t L_t + \tau_{i,t} K_{t+1} = 0. \tag{14}$$

From (14), one of the tax rates is determined by the other two rates. We choose $\tau_{w,t}$ and $\tau_{i,t}$ as independent variables, which are denoted by $\boldsymbol{\tau_t} = (\tau_{w,t}, \tau_{i,t})$.

Our goal is to design a tax policy that is time consistent. The timing of events in period t is as follows: (i)given k_t, the government sets $\boldsymbol{\tau_t}$ to maximize the household's utility; (ii)given the prices and tax rates, the household and firms make their decisions to maximize their own objectives; (iii)all markets clear, and l_t, c_t, and prices are determined; (iv)the values of τ_{rt} and k_{t+1} are determined from the budget constraints of the individual and the government. In other words, the government solves the problem in each period. Let us call the derived tax rates $\boldsymbol{\tau}_t$ the time-consistent tax policy.

Proposition 6. *The pair of constant tax rates $\overline{\boldsymbol{\tau}} = (\overline{\tau}_w, \overline{\tau}_i)$ is the time-consistent tax policy if they are given by*

$$\overline{\tau}_w = 0, \quad \overline{\tau}_i = \frac{\zeta}{1+\zeta}\left(\frac{1-\beta_c}{\beta_c}\frac{\beta_l}{1-\beta_l} - 1\right).$$

Proof. See the Appendix. □

Accordingly, $\overline{\tau}_r$ is given by $-\beta_c \overline{\tau}_i$. Since both the saving tax rate and the wage income tax rate turn out to be constant over time under the time-consistent tax policy, the saving rate and labor supply are given by the pair $(s^{eqm}(\overline{\boldsymbol{\tau}}), l^{eqm}(\overline{\boldsymbol{\tau}}))$.

Lemma 6. *In the market economy with a time-consistent tax policy $\overline{\boldsymbol{\tau}}$,*

1. $(s^{eqm}(\overline{\boldsymbol{\tau}}), l^{eqm}(\overline{\boldsymbol{\tau}})) = (s^{sp}, l^{sp})$;
2. $\overline{\tau}_i \lesseqgtr 0$ *if and only if* $\beta_c \gtreqless \beta_l$.

The proof of this lemma is given in the proof of Proposition 6. The first property means that under the time-consistent tax policy, $\overline{\boldsymbol{\tau}}$, the allocation by the social planner is replicated in the equilibrium. Then, the second property shows that the individual's savings must be subsidized (taxed) when $\beta_c > (<) \ \beta_l$. This result is intuitive given that, in the market economy, each individual's saving rate is excessively low (high) if she discounts future consumption at a lower (higher) rate than she does future leisure.

5 Concluding Remarks

We propose a dynamic model in which an individual's non-unitary time discounting induces preference reversals. We first characterize the market equilibrium in which the individual's decision-making satisfies time consistency. Then, from the normative point of view, we derive the following summarized results. First, a self in any period strictly prefers the social planning allocation to the laissez-faire allocation, provided the state variable has the same value. Therefore, our welfare properties differ from those of previous studies. Second, if we focus on the overall paths of the market equilibrium and social planning, the following two cases arise. If the individual discounts future leisure more steeply than she does future consumption, the planning allocation dominates the laissez-faire allocation in the Pareto sense. However, if she discounts future consumption more steeply than she does future leisure, a conflict can arise among the different selves of the individual.

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Appendix to
"Welfare Implications of Non-unitary Time Discounting"

A Proof of Proposition 1

We restate the optimization problem of the self in a period:

$$V(a) = \max_{c,x,a'} \left\{ u(c) + v(x) + \beta_c V_c(a') + \beta_x V_x(a') \;\middle|\; a' = Ra + w - c - px \right\}, \tag{A.1}$$

where V_c and V_x are recursively defined by the following functional equations:

$$V_c(a) = u(\phi_c(a)) + \beta_c V_c(g(a)), \quad V_x(a) = v(\phi_x(a)) + \beta_x V_x(g(a)). \tag{A.2}$$

The first-order conditions (FOCs) of the problem (A.1) are

$$\frac{v'(x)}{u'(c)} = p, \tag{A.3}$$

$$u'(c) = \beta_c \frac{dV_c(a')}{da'} + \beta_x \frac{dV_x(a')}{da'}. \tag{A.4}$$

Differentiating $V_c(a)$ and $V_x(a)$ in (A.2) with respect to a and adding these together, we obtain

$$\frac{d}{da}\left(V_c(a) + V_x(a)\right) = u'(c)\frac{d\phi_c(a)}{da} + v'(x)\frac{d\phi_x(a)}{da} + \left(\beta_c \frac{dV_c(a')}{da'} + \beta_x \frac{dV_x(a')}{da'}\right)\frac{dg(a)}{da}.$$

Substituting (A.3) and (A.4) into the right-hand side of this equation, we obtain

$$\frac{d}{da}\left(V_c(a) + V_x(a)\right) = u'(c)\left(\frac{d\phi_c(a)}{da} + p\frac{d\phi_x(a)}{da} + \frac{dg(a)}{da}\right). \tag{A.5}$$

Since $g(a)$ is defined as $g(a) \equiv Ra - \phi_c(a) - p\phi_x(a)$,

$$R = \frac{d\phi_c(a)}{da} + p\frac{d\phi_x(a)}{da} + \frac{dg(a)}{da}. \tag{A.6}$$

From (A.5) and (A.6), we obtain the following equation, which appears in the main body:

$$\frac{d}{da}\left(V_c(a) + V_x(a)\right) = u'(\phi_c(a))R. \tag{A.7}$$

Substituting (A.7) into (A.4) and evaluating it in period t yields the following Euler equation:

$$u'(c_t) = \beta_c u'(c_{t+1})R + (\beta_x - \beta_c)\frac{dV_x(a_{t+1})}{da_{t+1}}.$$

Using (A.3) and (A.7), the above equation is also expressed as

$$v'(x_t) = \beta_x v'(x_{t+1})R + (\beta_c - \beta_x)p\frac{dV_c(a_{t+1})}{da_{t+1}}.$$

B Proof of Lemma 1

To show this lemma, we guess $\phi_c(a, X)$ and $V_l(a, X)$ as follows:

$$\phi_c(a, X) = \mu_c R(X)a, \quad V_l(a, X) = b_l(X) + d_l \ln a, \tag{B.1}$$

where μ_c, $b_l(X)$, and d_l are the parameters that we have to solve for. From (7), we have

$$1 - \phi_l(a, X) = \frac{\zeta \mu_c R(X)a}{w(X)}. \tag{B.2}$$

Using this result and the budget constraint (2) in the main body, we obtain

$$a' = g(a, X) = \gamma R(X)a, \tag{B.3}$$

where

$$\gamma = 1 - (1 + \zeta)\mu_c. \tag{B.4}$$

Using the above guesses and (B.3), we can rewrite the Euler equation (8) as follows:

$$\begin{aligned} \frac{1}{\mu_c R(X)a} &= \beta_c \frac{1}{\mu_c a'} + (\beta_l - \beta_c)d_l \frac{1}{a'} \\ &= \left(\beta_c \frac{1}{\mu_c} + (\beta_l - \beta_c)d_l\right) \frac{1}{\gamma R(X)a}, \end{aligned}$$

which results in

$$\gamma = \beta_c + \mu_c(\beta_l - \beta_c)d_l. \tag{B.5}$$

At the same time, d_l (as well as $b_l(X)$) must satisfy the following functional equation:

$$V_l(a, X) = \zeta \ln(1 - \phi_l(a, X)) + \beta_l V_l(g(a, X), G(X))$$

Substituting (B.1)–(B.3) into the above equation yields

$$b_l(X) + d_l \ln a = \zeta \ln a + \beta_l d_l \ln a + \text{other terms}, \tag{B.6}$$

which results in $d_l = \zeta/(1 - \beta_l)$. Substituting this into (B.5),

$$\gamma = \beta_c + \zeta\mu_c \frac{\beta_l - \beta_c}{1 - \beta_l}. \tag{B.7}$$

From (B.4) and (B.7), we finally obtain

$$\gamma = \frac{\beta_c + \lambda}{1 + \lambda},$$

where

$$\lambda \equiv \frac{\zeta}{1 + \zeta} \frac{\beta_l - \beta_c}{1 - \beta_l}.$$

C Derivation of Equation (9)

When $\delta = 1$, human wealth in the main body is expressed as

$$\begin{aligned}\sum_{t'=t}^{\infty} \frac{w_{t'}}{\prod_{\nu=t}^{t'}(1+r_\nu-\delta)} &= \sum_{t'=t}^{\infty} \frac{w_{t'}}{\prod_{\nu=t}^{t'} r_\nu} \\ &= \sum_{t'=t}^{\infty} \left(\frac{1}{\prod_{\nu=t}^{t'-1} r_\nu} \frac{w_{t'}}{r_{t'}} \right). \end{aligned} \tag{C.1}$$

From equation (3),

$$\frac{w_{t'}}{r_{t'}} = \frac{1-\alpha}{\alpha} X_{t'}. \tag{C.2}$$

To simplify the expression of $\prod_{\nu=t}^{t'-1} r_\nu$, we first calculate $\sum_{\nu=t}^{t'-1} \ln r_\nu$.

$$\begin{aligned}\sum_{\nu=t}^{t'-1} \ln r_\nu &= \sum_{\nu=t}^{t'-1} \ln\left(A\alpha X_\nu^{\alpha-1}\right) \quad (\because \text{equation (3)}) \\ &= (\alpha-1)\sum_{\nu=t}^{t'-1} \ln X_\nu + (t'-t)\ln(A\alpha). \end{aligned} \tag{C.3}$$

Since we guess that $X_{t+1} = s^{eqm} A X_t^\alpha$ for all t,

$$\ln X_\nu = \ln(s^{eqm}A) + \alpha \ln X_{\nu-1} \Leftrightarrow \ln X_\nu = \alpha^{\nu-t}\ln X_t + \frac{1-\alpha^{\nu-t}}{1-\alpha}\ln(s^{eqm}A). \tag{C.4}$$

Then,

$$\begin{aligned}\sum_{\nu=t}^{t'-1} \ln X_\nu &= \left(\sum_{\nu=t}^{t'-1} \alpha^{\nu-t}\right)\ln X_t + \frac{1}{1-\alpha}\left[(t'-t) - \sum_{\nu=t}^{t'-1}\alpha^{\nu-t}\right]\ln(s^{eqm}A) \\ &= \frac{1-\alpha^{t'-t}}{1-\alpha}\ln X_t + \frac{1}{1-\alpha}\left[(t'-t) - \frac{1-\alpha^{t'-t}}{1-\alpha}\right]\ln(s^{eqm}A) \\ &= \frac{1}{1-\alpha}\left[\ln X_t - \ln X_{t'} + (t'-t)\ln(s^{eqm}A)\right], \end{aligned}$$

where we use the fact that $\ln X_{t'}$ satisfies equation (C.4) with $\nu = t'$. Substituting this into the right-hand side of equation (C.3) yields

$$\begin{aligned}\sum_{\nu=t}^{t'-1} \ln r_\nu &= -\ln X_t + \ln X_{t'} - (t'-t)\ln(s^{eqm}A) + (t'-t)\ln(A\alpha) \\ &= \ln X_{t'} - \ln X_t + (t'-t)\ln(\alpha/s^{eqm}). \end{aligned}$$

Since $\prod_{\nu=t}^{t'-1} r_\nu = \exp\left[\sum_{\nu=t}^{t'-1}\ln r_\nu\right]$, the above equation implies

$$\prod_{\nu=t}^{t'-1} r_\nu = \frac{X_{t'}}{X_t}\left(\frac{\alpha}{s^{eqm}}\right)^{t'-t}. \tag{C.5}$$

Finally, substituting (C.2) and (C.5) into (C.1), we obtain

$$\sum_{t'=t}^{\infty} \frac{1}{\prod_{\nu=t}^{t'-1} r_\nu} \frac{w_{t'}}{r_{t'}} = \sum_{t'=t}^{\infty} \frac{\frac{1-\alpha}{\alpha} X_{t'}}{\frac{X_{t'}}{X_t}\left(\frac{\alpha}{s^{eqm}}\right)^{t'-t}}$$

$$= \frac{1-\alpha}{\alpha} X_t \sum_{t'=t}^{\infty} \left(\frac{s^{eqm}}{\alpha}\right)^{t'-t}. \tag{C.6}$$

If $s^{eqm} \geq \alpha$, (C.6) implies that the human wealth of the household is infinity in any period, which should be excluded. Since we guess that $s^{eqm} < \alpha$ (and show that this inequality holds), equation (C.6) gives

$$\sum_{t'=t}^{\infty} \frac{1}{\prod_{\nu=t}^{t'-1} r_\nu} \frac{w_{t'}}{r_{t'}} = \frac{1-\alpha}{\alpha - s^{eqm}} X_t. \tag{C.7}$$

Since $X_t = k_t / l^{eqm}$ from the market-clearing conditions for labor and assets, we obtain equation (9).

D Proof of Lemma 2

We state the planner's problem again here.

$$V^{sp}(k) = \max_{k',l} \left\{ \ln(Ak^\alpha l^{1-\alpha} - k') + \zeta \ln(1-l) + \beta_c V_c^{sp}(k') + \beta_l V_l^{sp}(k') \right\}. \tag{D.1}$$

Assume that the planner in a period expects that if the value of capital is given by k, the next self's decisions about savings and labor supply are

$$k' = g^{sp}(k), \quad l = \phi_l^{sp}(k).$$

Then, functions $V_c^{sp}(k)$ and $V_l^{sp}(k)$ are given by the following functional equations:

$$V_c^{sp}(k) = \ln\left[Ak^\alpha(\phi_l^{sp}(k))^{1-\alpha} - g^{sp}(k)\right] + \beta_c V_c^{sp}(g^{sp}(k)), \tag{D.2}$$

$$V_l^{sp}(k) = \zeta \ln(1 - \phi_l^{sp}(k)) + \beta_l V_l^{sp}(g^{sp}(k)). \tag{D.3}$$

The FOCs of the problem in (D.1) with respect to k' and l are given by

$$\frac{1}{Ak^\alpha l^{1-\alpha} - k'} = \beta_c \frac{\partial V_c^{sp}(k')}{\partial k'} + \beta_l \frac{\partial V_l^{sp}(k')}{\partial k'}, \tag{D.4}$$

$$\frac{(1-\alpha)Ak^\alpha l^{-\alpha}}{Ak^\alpha l^{1-\alpha} - k'} - \zeta \frac{1}{1-l} = 0. \tag{D.5}$$

respectively. We make the following guess for V_i^{sp} ($i \in \{c, l\}$):

$$V_i^{sp}(k) = a_i + d_i \ln k.$$

From (D.4), we obtain

$$g^{sp}(k) = \frac{\Psi}{1+\Psi} Ak^\alpha l^{1-\alpha}, \tag{D.6}$$

where Ψ is defined as

$$\Psi \equiv \beta_c d_c + \beta_l d_l. \tag{D.7}$$

From (D.5) and (D.6), we obtain

$$\phi_l^{sp}(k) = l^{sp}, \qquad \text{where } l^{sp} = \frac{(1-\alpha)(1+\Psi)}{\zeta + (1-\alpha)(1+\Psi)}. \tag{D.8}$$

Since the value of Ψ is still unknown, we substitute (D.6), (D.8), and the guess for V_j^{sp} into (D.2) and (D.3):

$$\begin{aligned} a_c + d_c \ln k &= \ln\left(\frac{1}{1+\Psi} A k^{\alpha} (l^{sp})^{1-\alpha}\right) + \beta_c \left[a_c + d_c \ln\left(\frac{\Psi}{1+\Psi} A k^{\alpha} (l^{sp})^{1-\alpha}\right)\right], \\ a_l + d_l \ln k &= \zeta \ln(1-l^{sp}) + \beta_l \left[a_l + d_l \ln\left(\frac{\Psi}{1+\Psi} A k^{\alpha} (l^{sp})^{1-\alpha}\right)\right]. \end{aligned}$$

Since the coefficients of both sides must be equal, we can show that

$$d_c = \frac{\alpha}{1-\beta_c\alpha}, \qquad d_l = 0.$$

Substituting this result into the definition of Ψ, we have

$$\Psi = \frac{\beta_c\alpha}{1-\beta_c\alpha}.$$

Finally, substituting the obtained value of Ψ into (D.6) and (D.8), we obtain

$$l^{sp} = \frac{1-\alpha}{1-\alpha+\zeta(1-\beta_c\alpha)},$$

and

$$g^{sp}(k) = \beta_c\alpha A k^{\alpha}(l^{sp})^{1-\alpha} \Leftrightarrow s^{sp} = \beta_c\alpha.$$

E Proof of Proposition 3

If the saving rate and labor supply are determined so that they are constant over time, we calculate the utility of a self in period t as follows.

$$\begin{aligned} U_t &= \sum_{t'=t}^{\infty} \left\{\beta_c^{t'-t} \ln[(1-s)A(k_{t'})^{\alpha} l^{1-\alpha}] + \beta_l^{t'-t}\zeta \ln(1-l)\right\} \\ &= \frac{1}{1-\beta_c}\left[\ln(1-s) + \ln\left(A l^{1-\alpha}\right)\right] + \frac{\zeta}{1-\beta_l}\ln(1-l) + \sum_{t'=t}^{\infty}\beta_c^{t'-t}\alpha \ln k_{t'}. \end{aligned} \tag{E.1}$$

From equation (12) in the main body, we have

$$\ln k_t' = \alpha^{t'-t}\ln k_t + \frac{1-\alpha^{t'-t}}{1-\alpha}\ln(sAl^{1-\alpha}),$$

where k_t is historically given for the self in period t. Substituting this result into the last term in (E.1) yields

$$\begin{aligned} U_t &= \frac{1}{1-\beta_c}\ln(1-s) + \frac{\zeta}{1-\beta_l}\ln(1-l) + \frac{\alpha}{1-\beta_c\alpha}\ln k + \frac{\beta_c\alpha\ln s + \ln A + (1-\alpha)\ln l}{(1-\beta_c)(1-\beta_c\alpha)} \\ &= W(s,l,k). \end{aligned} \tag{E.2}$$

Note that the following identity holds:

$$V^i(k) \equiv W(s^i, l^i, k), \quad i \in \{eqm, sp\}.$$

To obtain this proposition, we first show the following two lemmas.

Lemma E.1. *There exists a unique pair* (s^*, l^*) *that maximizes* $W(s, l, k)$.

Proof. Since this function is strictly concave in (s, l), the necessary and sufficient conditions for (s^*, l^*) are given by the following FOCs:

$$\frac{\partial W}{\partial s} = 0: \quad \frac{1}{1-\beta_c}\frac{1}{1-s} = \frac{\beta_c \alpha}{(1-\beta_c)(1-\beta_c\alpha)}\frac{1}{s},$$
$$\frac{\partial W}{\partial l} = 0: \quad \frac{\zeta}{1-\beta_l}\frac{1}{1-l} = \frac{1-\alpha}{(1-\beta_c)(1-\beta_c\alpha)}\frac{1}{l},$$

which in turn yield

$$s^* = \beta_c \alpha,$$
$$l^* = \frac{1-\alpha}{1-\alpha+\zeta\omega(1-\beta_c\alpha)},$$

where $\omega \equiv (1-\beta_c)/(1-\beta_l)$, which deviates from unity if and only if $\beta_c \neq \beta_l$. □

We next show the following lemma.

Lemma E.2. $s^{sp} (\equiv s^*) \gtreqless s^{eqm}$ *and* $l^* \gtreqless l^{sp} \gtreqless l^{eqm}$ *if and only if* $\beta_c \gtreqless \beta_l$.

Proof. From their definitions, it follows that $(s^{eqm}, l^{eqm}) = (s^{sp}, l^{sp}) = (s^*, l^*)$ when $\beta_c = \beta_l$. From the proof of Lemma 3 in the main body, we know that s^{eqm} and l^{eqm} are strictly increasing with respect to β_l, whereas s^{sp} and l^{sp} are independent of β_l. Finally, l^* is strictly decreasing with respect to β_l. Then, we can show that

$$s^{sp} \equiv s^* \gtreqless s^{eqm} \Leftrightarrow \beta_c \gtreqless \beta_l,$$
$$l^* \gtreqless l^{sp} \gtreqless l^{eqm} \Leftrightarrow \beta_c \gtreqless \beta_l.$$

□

Having obtained Lemmas E.1 and E.2, we evaluate the ranking between $V^{eqm}(k)$ and $V^{sp}(k)$ by using the contours of $W(s, l, k)$ in the s-l plane. In panels (a) and (b) of Figure 1, (s^*, l^*) is located at point O, and the closed curves represent the contours of $W(s, l, k)$. The value of the welfare evaluation function $W(s, l, k)$ increases as the curves approach O. At any point on the curve passing through point A (B), $V^{eqm}(k)$ ($V^{sp}(k)$) is achieved. From Lemma E.2, the distance from points A to O is necessarily farther than that from points B to O, as long as $\beta_c \neq \beta_l$. Furthermore, from this lemma, we can show that points A and B are located in the same quadrant of the coordinate plane, with its origin given by point O. This means that when $\beta_c \neq \beta_l$, the indifference curve passing through point A is always located outside the curve passing through point B. Therefore, we show that this proposition holds.

F Proof of Lemma 5

From the definitions of $V^j(k)$ and W in (E.2), $V^j(k)$ is expressed as

$$V^j(k) = \frac{\alpha}{1-\beta_c\alpha}\ln k + A^j, \quad j \in \{eqm, sp\},$$

where A^j is a collection of other terms, independent of k. The above equation shows that $V^j(k)$ is strictly increasing.

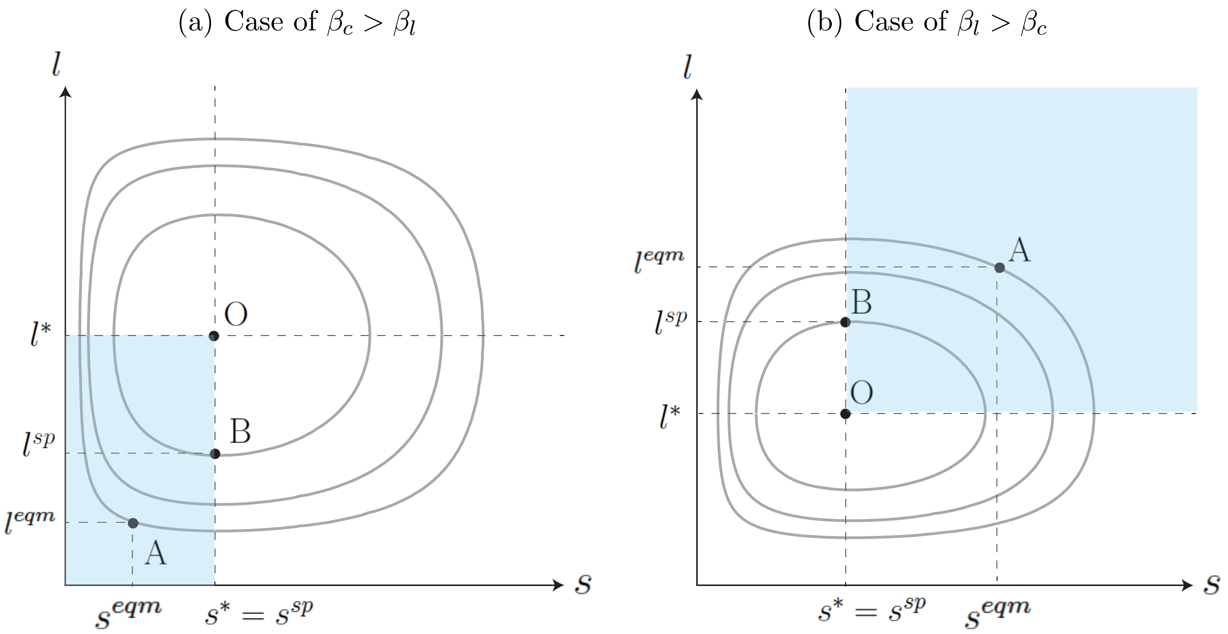

Figure 1: Contours of $W(s, l, k)$ and the interrelationship of (s^*, l^*), (s^{eqm}, l^{eqm}), and (s^{sp}, l^{sp})

G Proof of Proposition 5

When the saving rate and labor supply are constant over time, the steady state of capital is given by

$$k = K_{ss}(s, l) \equiv (sA)^{1/(1-\alpha)} l. \tag{G.1}$$

Thus, the steady state of k in the market equilibrium is $K_{ss}(s^{eqm}, l^{eqm})$, while that under social planning is $K_{ss}(s^{sp}, l^{sp})$. We define the function $W_{ss}(s, l)$ as follows:

$$\begin{aligned} W_{ss}(s, l) &\equiv W(s, l, K_{ss}(s, l)) \\ &= \frac{1}{1-\beta_c} \ln\left[(1-s)s^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} l\right] + \frac{\zeta}{1-\beta_l} \ln(1-l). \end{aligned}$$

We verify that $V^j(K_{ss}(s^j, l^j)) \equiv W_{ss}(s^j, l^j)$. By simple calculations, we find that $W_{ss}(s, l)$ is maximized at (s^*_{ss}, l^*_{ss}), where

$$s^*_{ss} = \alpha, \quad l^*_{ss} = \frac{1-\beta_l}{1-\beta_l + \zeta(1-\beta_c)}.$$

To obtain this proposition, we first show the following lemma.

Lemma G.1. *Given β_c, there exists a unique $\overline{\beta}_l \in (\beta_c, 1)$, such that $s^{sp} < s^{eqm} < s^*_{ss}$ and $l^{sp} < l^{eqm} < l^*_{ss}$ if and only if $\beta_c < \beta_l < \overline{\beta}_l$.*

Proof. From the proof of Lemma E.2, we show that

$$s^{sp} < s^{eqm}, \quad l^{sp} < l^{eqm} \;\Leftrightarrow\; \beta_l > \beta_c.$$

Then, our remaining task is to derive the condition under which $s^{eqm} < s^*_{ss}$ and $l^{eqm} < l^*_{ss}$ hold. From its definition, s^{eqm} converges to $\alpha (= s^*_{ss})$ as $\beta_l \to 1$. Since s^{eqm} is strictly increasing with respect to β_l, we first find that

$$s^{eqm} < s^*_{ss} \quad \text{for all } \beta_l \in (0, 1).$$

We next consider the ranking between l^{eqm} and l^*_{ss}. From its definition, l^{eqm} is strictly increasing with respect to β_l and converges to $1/(1+\zeta)$ as $\beta_l \to 1$. On the contrary, we can easily verify that l^*_{ss} is decreasing with respect to β_l, and $l^*_{ss} = 1/(1+\zeta)$ when $\beta_l = \beta_c$, and $l^*_{ss} \to 0$ as $\beta_l \to 1$. There exists a unique $\overline{\beta}_l \in (\beta_c, 1)$ such that

$$l^{eqm} < l^*_{ss} \;\Leftrightarrow\; \beta_l < \overline{\beta}_l.$$

These results show that this lemma holds. □

Lemma G.1 shows that if $\beta_c < \beta_l < \overline{\beta}_l$, $V^{eqm}(K_{ss}(s^{eqm}, l^{eqm})) > V^{sp}(K_{ss}(s^{sp}, l^{sp}))$. If $\beta_l > \overline{\beta}_l$, it is ambiguous which case yields higher welfare. Therefore, we focus on the former situation.

Since $V^{eqm}(k_0) < V^{sp}(k_0)$, there is at least one period, denoted by T, such that $V^{eqm}(k^{eqm}_t) > V^{sp}(k^{sp}_t)$ if $t \geq T$. Then, we can complete the proof of this proposition by showing that such a period T is unique. To this end, let q_t denote $q_t \equiv \ln k_t$. Then, we can rewrite (12) in the main body as follows:

$$\begin{aligned}(12) \Leftrightarrow \ln k^j_t &= \alpha^t \ln k_0 + (1-\alpha^t)\ln\left[(s^j A)^{1/(1-\alpha)} l^j\right] \\ \Leftrightarrow q^j_t &= \alpha^t q_0 + (1-\alpha^t) q^j_{ss}, \quad j \in \{eqm, sp\},\end{aligned} \tag{G.2}$$

where $q^j_{ss} \equiv \ln\left[(s^j A)^{1/(1-\alpha)} l^j\right] = \ln K_{ss}(s^j, l^j)$ from equation (G.1). Substituting (G.2) into $V^j(k)$, we obtain $V^j(k^j_t) \equiv \mathcal{V}^j(t)$, where $\mathcal{V}$ is given by

$$\mathcal{V}^j(t) \equiv \frac{\alpha}{1-\beta_c\alpha}[\alpha^t q_0 + (1-\alpha^t) q^j_{ss}] + A^j.$$

From Proposition 3, we know that $\mathcal{V}^{eqm}(0) \equiv V^{eqm}(k_0) < V^{sp}(k_0) \equiv \mathcal{V}^{sp}(0)$ always holds. Moreover, since we consider the case of $\beta_c < \beta_l < \overline{\beta}_l$, $\mathcal{V}^{eqm}(T) > \mathcal{V}^{sp}(T)$ as $T \to \infty$. Finally, subtracting $\mathcal{V}^{eqm}(t)$ from $\mathcal{V}^{sp}(t)$ yields

$$\mathcal{V}^{eqm}(t) - \mathcal{V}^{sp}(t) = \frac{\alpha(1-\alpha^t)(q^{eqm}_{ss} - q^{sp}_{ss})}{1-\beta_c\alpha} + A^{eqm} - A^{sp}.$$

Since the value of α^t decreases as t increases, $\mathcal{V}^{eqm}(t') - \mathcal{V}^{sp}(t') > \mathcal{V}^{eqm}(t) - \mathcal{V}^{sp}(t)$, for all $t' > t$, if $q^{eqm}_{ss} - q^{sp}_{ss} > 0$. Note that this condition is automatically satisfied for the case of $\beta_l > \beta_c$. Thus, there exists a unique $T^* > 0$, such that $\mathcal{V}^{eqm}(t) > \mathcal{V}^{sp}(t)$ if and only if $t \geq T^*$. This proves Proposition 5.

H Analysis of Tax Policies

H.1 Market Equilibrium under the Time-invariant Tax Policy

To obtain the time-consistent tax policy in Proposition 6, we first characterize the market equilibrium when the tax rates are constant over time: $\boldsymbol{\tau}_t = \boldsymbol{\tau}$ for all t, which is necessary to show this proposition. Let us introduce the following new functions:

$$\hat{r}(X) \equiv (1-\tau_r) r(X), \quad \hat{w}(X) \equiv (1-\tau_w) w(X),$$

where $r(X)$ and $w(X)$ are given by (3), and the new variable:

$$\hat{k}_{t+1} = (1+\tau_i) k_{t+1}.$$

The household's budget constraint is then expressed as

$$\hat{k}_{t+1} = \hat{r}(X_t) k_t + \hat{w}(X_t) l_t - c_t. \tag{H.1}$$

Lemma H.1. *In the market equilibrium with the constant tax policy, the saving rate, labor supply, and capital income tax rate are given by* $s^{eqm}(\boldsymbol{\tau})$, $l^{eqm}(\boldsymbol{\tau})$, *and* $\tau_r(\boldsymbol{\tau})$, *respectively, where*

$$s^{eqm}(\boldsymbol{\tau}) \equiv \frac{\psi(\alpha + (1-\alpha)\tau_w)}{\psi + (1+\zeta)(1+\tau_i)},$$
$$l^{eqm}(\boldsymbol{\tau}) \equiv \frac{(1-\tau_w)(1-\alpha)(1+\zeta+\psi)}{(1+\zeta)[\zeta(1+s^{eqm}(\boldsymbol{\tau})\tau_i) + (1-\tau_w)(1-\alpha)(1+\psi)]},$$
$$\tau_r(\boldsymbol{\tau}) \equiv -\frac{\tau_w(1-\alpha) + \tau_i s^{eqm}(\boldsymbol{\tau})}{\alpha},$$

and

$$\psi \equiv \frac{\beta_c}{1-\beta_c} + \zeta\frac{\beta_l}{1-\beta_l}.$$

Proof. As in the case of the laissez-faire environment in Section 3, the self in period t rationally expects the law of motion for the aggregate state X_t as equation (5) in the main body. Then, the optimization problem of the self in a period is given by

$$V(k,X) = \max_{c,l,\hat{k}'} \left\{ \ln c + \zeta\ln(1-l) + \beta_c V_c\left[\hat{k}'/(1+\tau_i), X'\right] + \beta_l V_l\left[\hat{k}'/(1+\tau_i), X'\right] \right\}, \tag{H.2}$$

subject to the budget constraint (H.1). Functions V_c and V_l here are defined as the following functional equations:

$$V_c(k,X) = \ln\phi_c(k,X) + \beta_c V_c\left[g(k,X)/(1+\tau_i), G(X)\right], \tag{H.3}$$
$$V_l(k,X) = \zeta\ln\phi_l(k,X) + \beta_l V_l\left[g(k,X)/(1+\tau_i), G(X)\right], \tag{H.4}$$

respectively, where $\phi_c(\cdot)$ and $\phi_l(\cdot)$ are the policy functions for c and l in this case, and $g(\cdot)$ is given by $g(\cdot) \equiv \hat{r}(\cdot)k + \hat{w}(\cdot)\phi_l(\cdot) - \phi_c(\cdot)$.

Hereafter, the arguments of the functions are omitted unless doing so would cause confusion. The FOCs of the problem (H.2) are

$$\frac{1+\tau_i}{c} = \beta_c\frac{\partial V_c}{\partial k'} + \beta_l\frac{\partial V_l}{\partial k'}, \tag{H.5}$$
$$\frac{\hat{w}}{c} = \frac{\zeta}{1-l}. \tag{H.6}$$

We derive the equilibrium in this case by use of "guess and verify." We guess V_i ($i \in \{c,l\}$) and G are given by

$$V_i = a_i + b_i\ln X + d_i\ln(k+\varphi X),$$
$$G = sAX^{\alpha}.$$

Then, from (H.1), (H.5), and (H.6), we obtain

$$l = \phi_l(k,X) \equiv 1 - \frac{\zeta}{1+\zeta+\Psi}\frac{\hat{r}}{\hat{w}}(k+\Lambda X), \tag{H.7}$$
$$c = \phi_c(k,X) \equiv \frac{\hat{r}}{1+\zeta+\Psi}(k+\Lambda X), \tag{H.8}$$
$$\hat{k}' = g(k,X) \equiv \frac{\Psi\hat{r}}{1+\zeta+\Psi}(k+\Lambda X) - (1+\tau_i)\varphi G, \tag{H.9}$$

where the definition of Ψ is the same as (D.7):

$$\Psi \equiv \beta_c d_c + \beta_l d_l,$$

and Λ is given by

$$\begin{aligned}\Lambda &\equiv \frac{\hat{w} + \varphi G(X)}{\hat{r}X} \\ &= \frac{(1-\tau_w)(1-\alpha) + (1+\tau_i)\varphi s}{(1-\tau_r)\alpha}.\end{aligned}$$

Substituting (H.7)–(H.9) into (H.3) and (H.4), we obtain

$$b_c = -\frac{1-\alpha}{(1-\beta_c\alpha)(1-\beta_c)}, d_c = \frac{1}{1-\beta_c}, b_l = -\frac{1}{1-\beta_l}, d_l = \frac{\zeta}{1-\beta_l},$$

implying that

$$\Psi = \psi \equiv \frac{\beta_c}{1-\beta_c} + \zeta\frac{\beta_l}{1-\beta_l}.$$

Meanwhile, because $\Lambda = \varphi$ holds, we have

$$\varphi = \frac{(1-\tau_w)(1-\alpha)}{(1-\tau_r)\alpha - (1+\tau_i)s}. \tag{H.10}$$

Now, we derive the equilibrium saving rate s^{eqm}, labor supply l^{eqm}, and capital income tax rate τ_r. Since $K_{t+1} = k_{t+1}$ in the equilibrium, the household's saving behavior must be consistent with the law of motion of aggregate capital in the market equilibrium. Recalling that $X_{t+1} = k_{t+1}/l_{t+1}$, this implies

$$G(k_t/l_t)l_{t+1} = (1+\tau_i)^{-1}g(k_t, k_t/l_t) \quad \forall k, l. \tag{H.11}$$

Using the household's condition (H.9), the guess $G(k/l) = sA(k/l)^\alpha$, and the guess that labor supply is constant in the equilibrium, we can rewrite this consistency condition (H.11) as

$$sl = \frac{\psi\alpha(1-\tau_r)}{(1+\tau_i)(1+\zeta+\psi)}(l+\varphi) - \varphi s,$$

which in turn yields

$$s = \frac{1-\tau_r}{1+\tau_i}\frac{\psi\alpha}{1+\zeta+\psi}. \tag{H.12}$$

Since τ_r is still to be determined, we use the government's budget constraint (equation (14) in the main body) and $X' = sAX^\alpha$ to obtain

$$\tau_r = -\frac{\tau_w(1-\alpha) + \tau_i s}{\alpha}. \tag{H.13}$$

Substituting (H.13) into (H.12), we obtain the equilibrium saving rate:

$$s = s^{eqm}(\boldsymbol{\tau}) \equiv \frac{\psi(\alpha + (1-\alpha)\tau_w)}{\psi + (1+\zeta)(1+\tau_i)}.$$

Then, substituting this result back into (H.10) and (H.13) yields

$$\begin{aligned}\varphi &= \varphi^{eqm}(\boldsymbol{\tau}) \equiv \frac{(1-\tau_w)(1-\alpha)}{\alpha + \tau_w(1-\alpha) - s^{eqm}(\boldsymbol{\tau})}, \\ \tau_r &= \tau_r(\boldsymbol{\tau}) \equiv -\frac{\tau_w(1-\alpha) + \tau_i s^{eqm}(\boldsymbol{\tau})}{\alpha}.\end{aligned}$$

Finally, we obtain the equilibrium labor supply l^{eqm} using the consistency condition $l^{eqm} = \phi_l(k, k/l^{eqm})$. From (H.7), this condition is rewritten as

$$l = \frac{(1-\tau_w)(1-\alpha)(1+\zeta+\psi) - \zeta(1-\tau_r)\alpha\varphi}{(1-\tau_w)(1-\alpha)(1+\zeta+\psi) + \zeta(1-\tau_r)\alpha}.$$

Then, substituting $\varphi^{eqm}(\boldsymbol{\tau})$ and $\tau_r(\boldsymbol{\tau})$ into this equation and rearranging the terms, we obtain

$$l = l^{eqm}(\boldsymbol{\tau}) = \frac{(1-\tau_w)(1-\alpha)(1+\zeta+\psi)}{(1+\zeta)[\zeta(1+s^{eqm}(\boldsymbol{\tau})\tau_i) + (1-\tau_w)(1-\alpha)(1+\psi)]}.$$

□

H.2 Definition of the Time-consistent Policy

In this section, we formally define the time-consistent tax policy. Suppose that the government in period t sets $\boldsymbol{\tau}_t = \widetilde{\boldsymbol{\tau}} \equiv (\widetilde{\tau}_w, \widetilde{\tau}_i)$. while the government in the other periods set the tax rates as $\overline{\boldsymbol{\tau}} = (\overline{\tau}_w, \overline{\tau}_i)$. If $\widetilde{\boldsymbol{\tau}} \neq \overline{\boldsymbol{\tau}}$, there is unilateral deviation of the government in period t. Let $\widetilde{G}(X, \widetilde{\boldsymbol{\tau}})$ denote the law of motion of X, which differs from $G(X)$ obtained in the previous section because of the current government's one-period deviation. By definition, $\widetilde{G}(X, \overline{\boldsymbol{\tau}}) \equiv G(X)$ (i.e., if $\widetilde{\boldsymbol{\tau}} = \overline{\boldsymbol{\tau}}$, they are the same function).

Then, the optimization problem of the household in period t (H.2) is replaced by

$$\widetilde{V}(k, X, \widetilde{\boldsymbol{\tau}}) = \max_{c,l,\hat{k}'} \left\{ \ln c + \zeta \ln(1-l) + \beta_c V_c \left[(1+\widetilde{\tau}_i)^{-1}\hat{k}', \widetilde{G}(X, \widetilde{\boldsymbol{\tau}}) \right] + \beta_l V_l \left[(1+\widetilde{\tau}_i)^{-1}\hat{k}', \widetilde{G}(X, \widetilde{\boldsymbol{\tau}}) \right] \right\}. \tag{H.14}$$

In equation (H.14), functions V_c and V_l are the same as (H.3) and (H.4), respectively, meaning that the household's decision-making is qualitatively the same as that in the previous section. This is simply because each individual makes her decision taking the factor prices and taxes as given.

Next, we consider the government's decision-making in period t. In contrast to the household's behavior, the government recognizes that it can affect the equilibrium labor supply. In what follows, we let $\widetilde{l}^{eqm}(\widetilde{\boldsymbol{\tau}})$ denote the equilibrium labor supply in period t. Owing to the current government's deviation, $\widetilde{l}^{eqm}(\widetilde{\boldsymbol{\tau}})$ can be a different function from $l^{eqm}(\overline{\boldsymbol{\tau}})$.

We can now define the time-consistent tax policy.

Definition 1. *The sequence $\{\boldsymbol{\tau}_t\}_{t=0}^{\infty}$, with $\boldsymbol{\tau}_t = \overline{\boldsymbol{\tau}}$ $\forall t = 0, 1, 2 \ldots$, is the time-consistent tax policy if*

$$\forall k_t > 0, \forall t = 0, 1, 2 \ldots, \quad \overline{\boldsymbol{\tau}} = \arg\max_{\widetilde{\boldsymbol{\tau}}} \widetilde{V} \left[k_t, \frac{k_t}{\widetilde{l}^{eqm}(\widetilde{\boldsymbol{\tau}})}, \widetilde{\boldsymbol{\tau}} \right].$$

In other words, the sequence of tax rates $\{\boldsymbol{\tau}_t\}_{t=0}^{\infty}$, with $\boldsymbol{\tau}_t = \overline{\boldsymbol{\tau}}$ $\forall t = 0, 1, 2 \ldots$ is the time-consistent tax policy if any selves of the government cannot obtain a strictly positive welfare gain by these selves' unilateral one-shot deviation from $\overline{\boldsymbol{\tau}}$.

H.3 Proof of Proposition 6

We now show Proposition 6. Suppose that the government in period t deviates from $\overline{\boldsymbol{\tau}}$ and sets $\boldsymbol{\tau}_t = \widetilde{\boldsymbol{\tau}}$. We guess that the law of motion of the aggregate state in this period is given by

$$G(X_t) = \widetilde{s}_t A X_t^{\alpha},$$

where $\widetilde{s}_t$ is the equilibrium saving rate in period t, which can differ from $s^{eqm}(\overline{\boldsymbol{\tau}})$ by the government's deviation in this period. Since the pair of tax rates after this period is always given by $\overline{\boldsymbol{\tau}}$, the equilibrium labor supply l_{t+j} $(j = 1, 2, \dots)$ is $l^{eqm}(\overline{\boldsymbol{\tau}})$. Therefore, equation (H.11) now implies

$$\begin{aligned} & G(k/l)l^{eqm}(\overline{\boldsymbol{\tau}}) = (1+\tau_i)^{-1}g(k, k/l) \ \forall k, l \\ \Leftrightarrow \quad & \widetilde{s}_t l^{eqm}(\overline{\boldsymbol{\tau}}) = \frac{\psi\alpha(1-\tau_r)}{(1+\tau_i)(1+\zeta+\psi)}(l+\widetilde{\varphi}_t) - \widetilde{\varphi}_t\widetilde{s}_t, \end{aligned} \tag{H.15}$$

where $\widetilde{\varphi}_t$ is the value of φ in period t. In addition, by imposing $\boldsymbol{\tau} = \widetilde{\boldsymbol{\tau}}$ and $l_{t+1} = l^{eqm}(\overline{\boldsymbol{\tau}})$ on the government budget constraint (14), we obtain

$$\begin{aligned} & \widetilde{\tau}_r r(X_t)k_t + \widetilde{\tau}_w w(X_t)l_t + \widetilde{\tau}_i G(X_t)l_{t+1} = 0 \\ \Leftrightarrow \quad & \widetilde{\tau}_r\alpha + \widetilde{\tau}_w(1-\alpha) + \widetilde{\tau}_i\widetilde{s}_t\ l^{eqm}(\overline{\boldsymbol{\tau}}) = 0. \end{aligned} \tag{H.16}$$

From (H.10), (H.15), (H.16), and the consistency condition, $\phi_l(k, k/l) = l$, the values of $\widetilde{s}_t$, l_t, $\widetilde{\varphi}_t$, and $\widetilde{\tau}_r$ are determined as the functions of both $\widetilde{\boldsymbol{\tau}}$ and $\overline{\boldsymbol{\tau}}$. Let their values be denoted by $\widetilde{s}^{eqm}(\widetilde{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}})$, $\widetilde{l}^{eqm}(\widetilde{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}})$, $\widetilde{\varphi}^{eqm}(\widetilde{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}})$, and $\widetilde{\tau}_r(\widetilde{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}})$, respectively. Then,

$$\begin{aligned} k_{t+1} &= \widetilde{s}^{eqm}(\widetilde{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}})Ak_t^{\alpha}(\widetilde{l}^{eqm}(\widetilde{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}))^{1-\alpha}, \\ c_t &= (1-\widetilde{s}^{eqm}(\widetilde{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}))Ak_t^{\alpha}(\widetilde{l}^{eqm}(\widetilde{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}))^{1-\alpha}. \end{aligned}$$

Now, we derive the welfare in period t, $\widetilde{V}[k_t, k_t/\widetilde{l}^{eqm}(\cdot), \widetilde{\boldsymbol{\tau}}]$. Since the value of φ is given by $\varphi^{eqm}(\overline{\boldsymbol{\tau}})$ in the subsequent periods, the following equation holds:

$$k_{t+1} + \varphi X_{t+1} = \left(1 + \frac{\varphi^{eqm}(\overline{\boldsymbol{\tau}})}{l^{eqm}(\overline{\boldsymbol{\tau}})}\right)k_{t+1}.$$

Substituting these results into equation (H.14) yields

$$\begin{aligned} \widetilde{V}\left[k_t, \frac{k_t}{\widetilde{l}^{eqm}(\widetilde{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}})}, \widetilde{\boldsymbol{\tau}}\right] = & \ln c_t + \zeta\ln\left(1-\widetilde{l}^{eqm}\right) + \beta_c\left(b_c+d_c\right)\ln\left[\widetilde{s}^{eqm}(\widetilde{\boldsymbol{\tau}})Ak_t^{\alpha}(\widetilde{l}^{eqm}(\widetilde{\boldsymbol{\tau}}))^{1-\alpha}\right] \\ & + \text{other terms}, \end{aligned}$$

where "other terms" represent the collection of all terms independent of $\widetilde{\boldsymbol{\tau}}$. Note that in the proof of Lemma H.1, the values of b_c and d_c are already given and $b_l + d_l = 0$ is already verified.

From the above equation, we find that $\widetilde{\boldsymbol{\tau}}$ affects $\widetilde{V}$ only through $\widetilde{s}^{eqm}$ and $\widetilde{l}^{eqm}$. This implies that obtaining $\widetilde{V}$ by choosing $\widetilde{\boldsymbol{\tau}}$ can also be achieved directly by choosing $\widetilde{s}^{eqm}$ and $\widetilde{l}^{eqm}$. The FOCs for s and l are given by

$$\begin{aligned} \frac{\beta_c(b_c+d_c)}{s} &= \frac{1}{1-s}, \\ \frac{(1-\alpha)[1+\beta_c(b_c+d_c)]}{l} &= \frac{\zeta}{1-l}. \end{aligned}$$

respectively. Letting $(\overline{s}, \overline{l})$ denote the solutions, we explicitly obtain

$$\overline{s} = \beta_c\alpha, \quad \overline{l} = \frac{1-\alpha}{1-\alpha+\zeta(1-\beta_c\alpha)}. \tag{H.17}$$

The tax policy $\overline{\boldsymbol{\tau}}$ is then determined from

$$\overline{s} = \widetilde{s}^{eqm}(\overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}), \quad \overline{l} = \widetilde{l}^{eqm}(\overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}).$$

Since $\widetilde{s}^{eqm}(\boldsymbol{\tau}, \boldsymbol{\tau}) = s^{eqm}(\boldsymbol{\tau})$, and $\widetilde{l}^{eqm}(\boldsymbol{\tau}, \boldsymbol{\tau}) = l^{eqm}(\boldsymbol{\tau})$, for all $\boldsymbol{\tau}$, substituting $s^{eqm}(\boldsymbol{\tau})$ and $l^{eqm}(\boldsymbol{\tau})$ in Lemma H.1 into (H.17) yields the following two equations:

$$\begin{aligned}
&\overline{s} = s^{eqm}(\boldsymbol{\tau}) \Leftrightarrow \beta_c\alpha = \frac{\psi(\alpha + (1-\alpha)\tau_w)}{\psi + (1+\zeta)(1+\tau_i)}, \\
&\overline{l} = l^{eqm}(\boldsymbol{\tau}) \Leftrightarrow \frac{1-\alpha}{1-\alpha+\zeta(1-\beta_c\alpha)} = \frac{(1-\tau_w)(1-\alpha)(1+\zeta+\psi)}{(1+\zeta)[\zeta(1+\beta_c\alpha\tau_i) + (1-\tau_w)(1-\alpha)(1+\psi)}.
\end{aligned}$$

The former and latter equations yield

$$1 + \tau_i = \frac{\psi}{(1+\zeta)\beta_c}\left[1 - \beta_c + \tau_w(1-\alpha)/\alpha\right],$$

and

$$1 + \tau_i = \frac{(1-\beta_c)\psi}{(1+\zeta)\beta_c} - \frac{\tau_w}{\beta_c}\left[\frac{1-\beta_c\alpha}{\alpha} + \frac{(1-\beta_c)\psi}{1+\zeta}\right],$$

respectively. Then, from the above two equations, we have $\overline{\tau}_w = 0$ and $\overline{\tau}_i = \frac{1-\beta_c}{(1+\zeta)\beta_c}\psi - 1$. From the definition of ψ, we then have

$$\overline{\tau}_i = \frac{\zeta}{1+\zeta}\left(\frac{1-\beta_c}{\beta_c}\frac{\beta_l}{1-\beta_l} - 1\right).$$

Finally, substituting $\overline{\tau}_w = 0$ and $s^{eqm}(\overline{\boldsymbol{\tau}}) = \beta_c\alpha$ into (H.13), we obtain $\overline{\tau}_r = -\beta_c\overline{\tau}_i$.